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MULTIPLICITIES OF EIGENVALUES FOR THE LAPLACE OPERATOR ON A SQUARE

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Abstract. We establish a formula for multiplicities of eigenvalues for the Laplace operator subject to the Dirichlet boundary condition on a square. In particular, we show that for any given positive integer m, there is an eigenvalue whose multiplicity is exactly m.

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1. Introduction

Pythagoras of Samos was an ancient Greek philosopher and mathematician best known for his contributions to geometry. He is also credited with the discovery that strings whose lengths have a ratio of small integers produce harmonious sounds. This discovery is the foundation of Pythagorean tuning in music. Although Pythagoras could not have known this, his theory can be explained using the language of partial differential equations. Suppose we have a homogeneous elastic string of length L with ends tied along the horizontal x-axis at $x = 0$ and $x = L$. Let $u(x, t)$ be the displacement from equilibrium at position x and time t. Let T be the tension constant and ρ the mass density. From Newton's second law of motion applied to the string over the interval $[x,\ x + \Delta x]$, we obtain

$$T \cdot \frac{\partial u}{\partial x}(x + \Delta x, t) - T \cdot \frac{\partial u}{\partial x}(x, t) \approx \rho \Delta x \cdot \frac{\partial^2 u}{\partial t^2}.$$

Dividing both sides by Δx and letting $\Delta x \to 0$, we then arrive at the 1-dimensional wave equation with initial and boundary conditions:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}; \\ u(x, 0) = f(x), \quad \dfrac{\partial u}{\partial t}(x, 0) = g(x); \\ u(0, t) = u(L, t) = 0. \end{cases} \tag{1.1}$$

Here $f(x)$ is the initial position and $g(x)$ the initial velocity of the string. The constant $c = \sqrt{T/\rho}$ is called the *wave speed* of the vibration.

Solving this equation by the method of separation of variables, we set $u(x, t) = X(x)T(t)$. Then

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda,$$

where $\lambda \geq 0$ is a constant. Solving the boundary value problem

$$X'' = -\lambda X, \quad X(0) = X(L) = 0,$$

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we obtain the eigenvalues and the associated eigenfunctions

$$\lambda_k = \left(\frac{k\pi}{L}\right)^2, \qquad X_k(x) = \sin\left(\frac{k\pi x}{L}\right), \quad k = 1, 2, \dots. \tag{1.2}$$

From $T'' = -\lambda_k c^2 T$, we then have

$$T_k = A_k \cos(c\sqrt{\lambda_k} t) + B_k \sin(c\sqrt{\lambda_k} t).$$

It follows that the solution to the boundary value problem (1.1) is given by

$$u(x,t) = \sum_{k=1}^{\infty} \sin\left(\frac{k\pi x}{L}\right)\left(A_k \cos\left(\frac{kc\pi t}{L}\right) + B_k \sin\left(\frac{kc\pi t}{L}\right)\right), \tag{1.3}$$

where A_k and B_k are determined by the Fourier sine series of $f(x)$ and $g(x)$ over $[0,\ L]$ as follows:

$$A_k = \frac{2}{L}\int_0^L f(x)\sin\frac{k\pi x}{L}dx, \quad B_k = \frac{2}{kc\pi}\int_0^L g(x)\sin\frac{k\pi x}{L}\,dx.$$

From (1.3), we know that the fundamental frequency of vibration is

$$F_1 = \frac{c}{2L} = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

and the frequencies of the overtones are

$$F_k = \frac{kc}{2L} = \frac{k}{2L}\sqrt{\frac{T}{\rho}}, \quad k = 2, 3, \dots.$$

Notice that the frequencies of the overtones are integral multiples of the fundamental frequency. Relationship among amplitudes of the frequencies of the fundamental tone and the overtones determines the *timbre* of a music instrument, the characteristics of how it sounds. We refer the reader to [1] for an excellent exposition on the subject.

We now consider an elastic and homogeneous drumhead stretched over a frame. The frame is represented as a domain Ω in the xy-plane. Let $u(x,y,t)$ be the vertical displacement of the membrane from the equilibrium position and assume that the horizontal displacement is negligible. For any disk $D \subset\subset \Omega$, it follows from Newton's second law of motion that

$$\int_{\partial D} T\frac{\partial u}{\partial \mathbf{n}}dS = \int_D \rho u_{tt} dA,$$

where T is the tension constant, ρ the density constant, and $\mathbf{n}$ the outward normal direction of the boundary ∂D of the domain D. By the divergence theorem, we then have

$$\int_D T\Delta u dA = \int_D \rho u_{tt} dA,$$

where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplace operator. Dividing both sides by the area of the disk and letting its radius tend to 0, we then arrive at the wave equation:

$$u_{tt} = c^2\Delta u \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{1.4}$$

where $c = \sqrt{T/\rho}$.

Solving this wave equation by separation of variables, we let $u(x,y,t) = T(t)V(x,y)$ where $T(t)$ is a function depending only on the time variable t and $V(x,y)$ is a function depending only on the spatial variables x and y. It then follows from (1.4) that

$$\frac{T''}{c^2T} = \frac{\Delta V}{V} = -\lambda,$$

where λ is a constant. The boundary value problem (1.4) is now reduced to solving the following Helmholtz equation subject to the Dirichlet boundary condition:

$$\Delta V = -\lambda V \text{ on } \Omega, \quad V = 0 \text{ on } \partial\Omega. \tag{1.5}$$

The λ's are the eigenvalues of the Dirichlet Laplacian, the (negative of the) Laplace operator subject to the Dirichlet boundary condition.

It follows from Rellich's compactness lemma that the spectrum of the Dirichlet Laplacian consists of isolated eigenvalues of finite multiplicity (see, *e.g.*, [6, Theorem 6.2.3]). Let

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \leq \ldots$$

be the eigenvalues, arranged in an increasing order and repeated according to multiplicity. Let $\varphi_k(x,y)$ be the eigenfunction associated with the eigenvalue λ_k. Then the solution to the wave equation (1.4) has the form of

$$u(x,y,t) = \sum_{k=1}^{\infty} \left(A_k \cos(c\sqrt{\lambda_k}t) + B_k \sin(c\sqrt{\lambda_k}t) \right) \varphi_k(x,y).$$

The terms in the summation have frequencies

$$F_k = \frac{c\sqrt{\lambda_k}}{2\pi}, \quad k = 1, 2, \ldots,$$

in the time variable. F_1 is the fundamental frequency while the F_j's, $j \geq 2$, are the frequencies of the overtones of the drum. The above formula tells us that the frequencies of the vibration are obtained by multiplying the square root of the eigenvalues by a constant.

2. Multiplicities of Eigenvalues

The multiplicity mult (λ) of an eigenvalue λ is the number of linearly independent eigenfunctions associated to the eigenvalue λ:

$$\text{mult}\,(\lambda) = \dim\{V \mid -\Delta V = \lambda V \text{ on } \Omega, \quad V = 0 \text{ on } \partial\Omega\}.$$

Physically, the multiplicity represents the number of modes associated to the same frequency. In this paper, we study the structure of the set of all multiplicities and how it is related to geometry of the domain. We are particularly interested in characterizing domains that satisfy the following

Property (M): A domain is said to satisfy Property (M) if for any positive integer n, there exists an eigenvalue λ such that mult $(\lambda) = n$.

For the vibration of a string, it follows from (1.2) that all eigenvalues are simple (*i.e.*, they all have multiplicity one). The situation is not as simple in higher dimensions. The classical Courant's Nodal Domain Theorem states that the number of nodal domains of an eigenfunction associated to the k^{th}-eigenvalue is at most k. (Recall that the nodal domains are the connected components of the complement in Ω of the zero set of the eigenfunction.) As a consequence, the first eigenvalue on a (connected) domain is always simple. Determining multiplicities of higher eigenvalues is a highly non-trivial problem. For smoothly bounded planar domains, S.-Y. Cheng [3] showed that mult $(\lambda_2) \leq 3$ and

Nadirashvili [10] showed that $\operatorname{mult}(\lambda_k) \le 2k-1$ for $k \ge 3$ (see the recent preprint [2] and references therein for an extensive discussion of relevant results).

In general, it is difficult to explicitly compute the eigenvalues and determine their multiplicities on a planar domain. Explicit formulas for the eigenvalues are known only for a few cases such as circles, rectangles, equilateral triangles, hemi-equilateral triangles, and isosceles right triangles (see [9]).

The eigenvalues on the rectangle

$$R_{a,b} = \{(x,y) \mid 0 \le x \le a,\, 0 \le y \le b\}, \quad a,b > 0,$$

can be computed by using separation of variables $V(x,y) = X(x)\cdot Y(y)$. These eigenvalues and their associated eigenfunctions are

$$\lambda_{m,n} = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right),$$
$$V_{m,n}(x,y) = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$

for $m, n \in \mathbb{N}$, where $\mathbb{N}$ is the set of all positive integers. For any $\lambda > 0$, the multiplicity of λ is given by

$$\operatorname{mult}(\lambda) = \#\{(m,n) \in \mathbb{N}\times\mathbb{N} \mid \frac{m^2}{(a\sqrt{\lambda}/\pi)^2} + \frac{n^2}{(b\sqrt{\lambda}/\pi)^2} = 1\}.$$

Note that we have used the convention that $\operatorname{mult}(\lambda) = 0$ if λ is not an eigenvalue. Thus $\operatorname{mult}(\lambda)$ is the number of integer lattice points on the ellipse with semiaxes $a\sqrt{\lambda}/\pi$ and $b\sqrt{\lambda}/\pi$ in the first quadrant.

For convenience, we say the rectangle $R_{a,b}$ is *rational* if $(a/b)^2$ is rational; otherwise, we say it is *irrational.*

Proposition 2.1. *On irrational rectangles,*

$$\operatorname{mult}(\lambda_{m,n}) = 1, \quad \forall m,n \in \mathbb{N}.$$

Proof. Suppose otherwise, then there are $(m_1,n_1) \ne (m_2,n_2)$ such that

$$\lambda = \pi^2\left(m_1^2/a^2 + n_1^2/b^2\right) = \pi^2\left(m_2^2/a^2 + n_2^2/b^2\right).$$

Then

$$(b/a)^2 = (n_2^2 - n_1^2)/(m_1^2 - m_2^2).$$

This contradicts the assumption that $(a/b)^2$ is irrational. □

3. The Case of Squares

On a square with side length a, $\operatorname{mult}(\lambda)$ is the number of integer lattice points on the circle with radius $a\sqrt{\lambda}/\pi$ in the first quadrant. The problem of counting integer lattice points on a circle has a long history, dating back to Pythagoras, Fermat, Euler, Legendre, Gauss, and others (see [4]).

Let $r_2(k)$ be the number of ways k can be expressed as a sum of squares of a pair of (ordered) integers. Namely, $r_2(k)$ is the number of integer lattice points on the circle with radius $\sqrt{k}$. For $\lambda > 0$, set $k = a^2\lambda/\pi^2$. Taking into account of symmetry and the fact that lattice points on the coordinate axes are excluded, we then have

$$\operatorname{mult}(\lambda) = \begin{cases} r_2(k)/4, & k \text{ is not a square;} \\ (r_2(k)-4)/4, & k \text{ is a square.} \end{cases}$$

To compute $r_2(k)$, we use a formula due to Legendre (see [5, Theorem 2.2.11]). Let $d_1(k)$ be the number of divisors of k which are congruent to $1 \mod 4$ and $d_3(k)$ the number of divisors of k which are congruent to $3 \mod 4$. Legendre's formula states that for any positive integer k,

$$r_2(k) = 4\left(d_1(k) - d_3(k)\right).$$

With Legendre's formula, we then obtain the main result of this paper:

Theorem 3.1. *On a square with side length a, for any $\lambda > 0$, set $k = a^2\lambda/\pi^2$. Then*

$$\operatorname{mult}(\lambda) = \begin{cases} d_1(k) - d_3(k), & k \text{ is not a square;} \\ d_1(k) - d_3(k) - 1, & k \text{ is a square;} \\ 0, & k \text{ is not an integer.} \end{cases}$$

We now examine whether squares satisfy Property (M). For any positive integer n, we need to find an eigenvalue λ whose multiplicity is n. Let p be any prime that is congruent to $1 \mod 4$. Then p^{n-1} is a square if n is odd, while for even n it is not (because $\sqrt{p}$ is irrational). Thus the above theorem gives

$$\operatorname{mult}(p^{n-1}\pi^2/a^2) = \begin{cases} n, & n \text{ is even,} \\ n-1, & n \text{ is odd.} \end{cases}$$

The above construction gives a sequence of eigenvalues $\lambda = p^{n-1}\pi^2/a^2$ whose multiplicities include all *even* positive integers. To obtain a sequence of eigenvalues with multiplicities that include *all* positive integers, we set $\lambda = 2 \cdot p^{n-1}\pi^2/a^2$. The factor 2 is introduced so that $2 \cdot p^{n-1}$ is never a square regardless of whether n is even or odd. From the above theorem, we then have

$$\operatorname{mult}(2 \cdot p^{n-1}\pi^2/a^2) = n,$$

for any positive integer n. We have thus shown that squares satisfy Property (M).

We now turn to isosceles right triangles. Let $T_a = \{(x, y) \mid 0 < x < a, 0 < y < x\}$ be the isosceles right triangle with side length a. By reflecting the triangle about its hypotenuse and using the above computations on the resulting square, we obtain the eigenvalues and the associated eigenfunctions:

$$\lambda_{mn} = \frac{\pi^2}{a^2}(m^2 + n^2),\ u_{mn} = \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{a} - \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{a}, \quad m, n \in \mathbb{N}, n > m.$$

Thus for any $\lambda > 0$, $\operatorname{mult}(\lambda)$ is the number of integer lattice points (m, n) on the circle with radius $a\sqrt{\lambda}/\pi$ in the first quadrant above the line $y = x$. Note that the lattice point (m, n) on the line $y = x$ corresponds to $k = m^2 + n^2$ being the double of a square. From Theorem 3.1, we then have in this case

$$\operatorname{mult}(\lambda) = \begin{cases} \frac{1}{2}(d_1(k) - d_3(k)), & k \text{ is not a square nor a double of a square;} \\ \frac{1}{2}(d_1(k) - d_3(k) - 1), & k \text{ is a square or a double of a square;} \\ 0, & k \text{ is not an integer,} \end{cases}$$

where $k = a^2\lambda/\pi^2$ as before. Thus for the right isosceles triangle with side length a,

$$\operatorname{mult}(p^{n-1}\pi^2/a^2) = \operatorname{mult}(2 \cdot p^{n-1}\pi^2/a^2) = \begin{cases} n/2, & n \text{ is even;} \\ (n-1)/2, & n \text{ is odd.} \end{cases}$$

It follows that isosceles right triangles also satisfy Property (M).

4. Further Remarks

(1) We have shown that squares and isosceles right triangles satisfy Property (M) while irrational rectangles do not. The calculation for multiplicities of eigenvalues for a general *rational* rectangle is more complicated. It requires deeper results from number theory and will be studied in a forthcoming paper.

(2) Eigenvalues of the Dirichlet Laplacian on a circle can be expressed in terms of the zeros of the Bessel functions. It is well known that except for the first eigenvalue, all other eigenvalues on a circle have multiplicity 2 (see, *e.g.*, [7]).

(3) Eigenvalues on an equilateral triangle were explicitly computed by Gabriel Lamé in 1833. Multiplicities of these eigenvalues have been studied by McCartin, Pinsky, and others. We refer the reader to [9] for an extensive treatment on the subject. As will be shown in the forthcoming paper, equilateral triangles also satisfy Property (M).

(4) It would be interesting to characterize polygons that satisfy Property (M). The works on this problem for rectangles and equilateral triangles have already demonstrated a fascinating connection among geometry, number theory, and partial differential equations.

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